K U W A I T   U N I V E R S I T Y
Department of Mathematics

Math 102 — 12/05/2011
Calculus (2) — Second Midterm — Time: 90 minutes

$Q1$. Evaluate the following integrals:

a. $\displaystyle\int \sqrt{x}\,\tan^{-1}\sqrt{x}\,dx$

b. $\displaystyle\int \sinh^3 x\,\cosh^2 x\,dx$

c. $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

d. $\displaystyle\int \frac{1}{x\,(1+\sqrt[3]{x})^2}\,dx$

e. $\displaystyle\int \frac{x}{(x-2)\sqrt{x^2-4x+3}}\,dx$

$Q2$. Show that the improper integral

$$\int_1^2 \frac{1}{x\sqrt{4-x^2}}\,dx$$

converges to $\ln\sqrt{2+\sqrt{3}}$.

$Q3$. Use integration by parts to evaluate the integral $\displaystyle\int f(x)\,dx$, where $\displaystyle f(x)=\int_1^x \frac{\sin t}{t}\,dt$.

$Q1 = 20$ points (4 points for each part)
$Q2 = 3$ points
$Q3 = 2$ points

Solutions

$\mathcal{Q}1.$

a. Integrate by parts: put $u = \tan^{-1}\sqrt{x}$ and $dv = \sqrt{x}\,dx$ to get

$$du = \frac{1}{2\sqrt{x}}\,\frac{1}{1+x}\,dx\,, \quad v = \frac{2}{3}\,x\,\sqrt{x} \quad \to \quad v\,du = \frac{1}{3}\,\frac{x}{1+x}\,dx$$

$$I = \frac{2}{3}\,x\,\sqrt{x}\,\tan^{-1}\sqrt{x} - \frac{1}{3}\int \frac{x}{1+x}\,dx = \frac{2}{3}\,x\,\sqrt{x}\,\tan^{-1}\sqrt{x} - \frac{1}{3}\,x + \frac{1}{3}\,\ln|1+x| + c$$

b. $\sinh^3 x\,\cosh^2 x = \sinh^2 x\,\cosh^2 x\,\sinh x = (\cosh^2 x - 1)\,\cosh^2 x\,\sinh x$ and put $u = \cosh x$

$$I = \int (u^4 - u^2)\,du = u^5/5 - u^3/3 + c = \cdots$$

c. There are at least two ways to do this. Here is one:

$$\int \frac{\tan^3 x}{\sec^{1/2} x}\,dx = \int \frac{\tan^2 x}{\sec^{3/2} x}\,\sec x\,\tan x\,dx = \int \frac{\sec^2 x - 1}{\sec^{3/2} x}\,\sec x\,\tan x\,dx = \int (u^{1/2} - u^{-3/2})\,du$$

$$I = \frac{2}{3}\,u^{3/2} + 2\,u^{-1/2} + c = \frac{2}{3}\,(\sec x)^{3/2} + 2\,(\cos x)^{1/2} + c$$

d. Put $x = u^3$ (that is $u = \sqrt[3]{x}$)

$$\int \frac{1}{x\,(1+\sqrt[3]{x})^2}\,dx = \int \frac{1}{u^3\,(1+u)^2}\,3u^2\,du = \int \frac{3}{u\,(1+u)^2}\,du = 3\,\ln\left|\frac{u}{1+u}\right| + \frac{3}{1+u} + c$$

$$\frac{3}{u\,(1+u)^2} = 3\left[\frac{1}{u} - \frac{1}{1+u} - \frac{1}{(1+u)^2}\right]$$

e. Write $x^2 - 4x + 3 = (x-2)^2 - 1$ and let $x - 2 = \sec\theta$

$$I = \int \frac{x}{(x-2)\,\sqrt{x^2 - 4x + 3}}\,dx = \int (2 + \sec\theta)\,d\theta = 2\theta + \ln|\sec\theta + \tan\theta| + c$$

Now use the little triangle to convert back to x:

$$I = 2\sec^{-1}(x-2) + \ln|x - 2 + \sqrt{x^2 - 4x + 3}| + c$$

$\mathcal{Q}2.$

The indefinite integral: with $x = 2\sin\theta$, it becomes

$$\int \frac{1}{x\,\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta}{2\sin\theta\,2\,\cos\theta}\,d\theta = \frac{1}{2}\int \csc\theta\,d\theta = \frac{1}{2}\,\ln|\csc\theta - \cot\theta| = \frac{1}{2}\,\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right|$$

The improper integral: $x = 2$ is the culprit

$$I = \int_1^2 \frac{1}{x\,\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-}\int_1^t \frac{1}{x\,\sqrt{4-x^2}}\,dx = \frac{1}{2}\,\lim_{t\to 2^-}\,\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| - \frac{1}{2}\,\ln|2-\sqrt{3}|$$

but the limit is zero and we end up with

$$I = -\frac{1}{2}\,\ln|2-\sqrt{3}| = \frac{1}{2}\,\ln\frac{1}{2-\sqrt{3}} = \frac{1}{2}\,\ln(2+\sqrt{3}) = \ln\sqrt{2+\sqrt{3}}$$

$\mathcal{Q}3.$

Let $u = f(x)$ and $dv = dx$. Then $du = \sin x\,dx/x$, $v = x$, and $v\,du = \sin x\,dx$. The rest is easy

$$\int f(x)\,dx = x\,f(x) - \int \sin x\,dx = x\,f(x) + \cos x + c$$